$$-82\sin^3\delta\cos\delta+32\sin^5\delta\cos\delta+96\sin(\theta-\delta)\sin\theta\sin\delta(\delta-\sin\delta\cos\delta)^2]\sin^2\theta\sin^2\delta d\theta d\delta$$

$$=\frac{5a^2}{162\pi^4}\int_0^\pi(360\theta^2-144\theta^3\sin\theta\cos\theta-360\theta^2\sin^4\theta+252\theta^2\sin^2\theta+336\theta\sin^5\theta\cos\theta$$

$$-132\theta\sin^3\cos\theta-615\theta\sin\theta\cos\theta+72\sin^8\theta-250\sin^6\theta-301\sin^4\theta+255\sin^2\theta)\sin^2\theta d\theta$$

$$=\frac{5a^2}{12\pi}\left(5-\frac{1001}{144\pi^2}\right).$$

MISCELLANEOUS.

116. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Prove that

$$-\left|\alpha^2\beta^2\gamma^2\right|^2\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2=\begin{Vmatrix} \begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & \gamma_1 \\ a & b & c & \gamma_2 \\ c & e & f & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \end{Vmatrix}$$

[From Muir's *Determinants*.]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x, y, z be the minors with respect to α_1, α_2, α_3 in the first row, u, v, w the minors with respect to β_1, β_2, β_3 in the second row, and r, s, t the minors with respect to γ_1, γ_2, γ_3 in the third row of the determinant $\triangle$ on the right hand side of the equality. Then

$$=-\begin{vmatrix} \alpha_1x-\alpha_2y+\alpha_3z, & \beta_1x-\beta_2y+\beta_3z, & \gamma_1x-\gamma_2y+\gamma_3z \\ \alpha_1u-\alpha_2v+\alpha_3w, & \beta_1u-\beta_2v+\beta_3w, & \gamma_1u-\gamma_2v+\gamma_3w \\ \alpha_1r-\alpha_2s+\alpha_3t, & \beta_1r-\beta_2s+\beta_3t, & \gamma_1r-\gamma_2s+\gamma_3t \end{vmatrix}$$

$$=\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_2 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}\begin{vmatrix} x & y & z \\ u & v & w \\ r & r & t \end{vmatrix}=(\alpha_1\beta_2\gamma_3)\begin{vmatrix} x & y & z \\ u & v & w \\ r, & s & t \end{vmatrix}$$

$$=(\alpha_1\beta_2\gamma_3)\begin{vmatrix} \begin{vmatrix} b & c & \alpha_1 \\ d & e & \alpha_2 \\ e & f & \alpha_3 \end{vmatrix} & \begin{vmatrix} a & c & \alpha_1 \\ b & e & \alpha_2 \\ c & f & \alpha_3 \end{vmatrix} & \begin{vmatrix} a & b & \alpha_1 \\ b & d & \alpha_2 \\ c & e & \alpha_3 \end{vmatrix} \\ \begin{vmatrix} b & c & \beta_1 \\ d & e & \beta_2 \\ e & f & \beta_3 \end{vmatrix} & \begin{vmatrix} a & c & \beta_1 \\ b & e & \beta_2 \\ c & f & \beta_3 \end{vmatrix} & \begin{vmatrix} a & b & \beta_1 \\ b & d & \beta_2 \\ c & e & \beta_3 \end{vmatrix} \\ \begin{vmatrix} b & c & \gamma_1 \\ d & e & \gamma_2 \\ e & f & \gamma_3 \end{vmatrix} & \begin{vmatrix} a & c & \gamma_1 \\ b & e & \gamma_2 \\ c & f & \gamma_3 \end{vmatrix} & \begin{vmatrix} a & b & \gamma_1 \\ b & d & \gamma_2 \\ c & e & \gamma_3 \end{vmatrix} \end{vmatrix}$$

Let A, B, C, etc., be the minors with respect to a, b, c, etc. Then

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2 = \begin{vmatrix} A & -B & C \\ -B & D & -E \\ C & -E & F \end{vmatrix}$$

$$\therefore\ \Delta=(\alpha_1\beta_2\gamma_3)\begin{vmatrix} \alpha_1A-\alpha_2B+\alpha_3C, & \alpha_1B-\alpha_2D+\alpha_3E, & \alpha_1C-\alpha_2E+\alpha_3F \\ \beta_1A-\beta_2B+\beta_3C, & \beta_1B-\beta_2D+\beta_3E, & \beta_1C-\beta_2E+\beta_3F \\ \gamma_1A-\gamma_2B+\gamma_3C, & \gamma_1B-\gamma_2D+\gamma_3E, & \gamma_1C-\gamma_2E+\gamma_3F \end{vmatrix}$$

$$=-(\alpha_1\beta_2\gamma_3)^2\begin{vmatrix} A & -B & C \\ -B & D & -E \\ C & -E & F \end{vmatrix}=-(\alpha_1\beta_2\gamma_3)^2\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2.$$

117. Proposed by **W. J. GREENSTREET, M. A.,** Editor of The Mathematical Gazette, Stroud, England.

If $x\cos\alpha+y\cos\alpha=a\cos\theta+b\cos\varphi$, and $x\sin\alpha+b\sin\varphi=y\sin\alpha+a\sin\theta=\kappa$, find the maximum value of κ, and the values of x and y.

Solution by **G. B. M. ZERR, A.M., Ph. D.,** Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$(x+y)\cos\alpha=a\cos\theta+b\cos\varphi....(1).$$
$$(x+y)\sin\alpha+a\sin\theta+b\sin\varphi=2\kappa....(2).$$

(1) in (2) gives

$$\frac{(a\cos\theta+b\cos\varphi)\sin\alpha}{\cos\alpha}+a\sin\theta+b\sin\varphi=2\kappa, \text{ or}$$

$$a\sin(\theta+\alpha)+b\sin(\varphi+\alpha)-2\kappa\cos\alpha=0=u.$$

$\therefore\ du/d\theta=a\cos(\theta+\alpha)=0,\ du/d\varphi=b\cos(\varphi+\alpha)=0.$

$\therefore\ \theta=\varphi=\frac{1}{2}\pi-\alpha$ for a maximum.

$\therefore\ \kappa=\frac{1}{2}(a+b)\sec\alpha$ is the maximum value.

$$x\cos\alpha+\frac{(\kappa-a\sin\theta)\cos\alpha}{\sin\alpha}=a\cos\theta+b\cos\varphi.$$